

Hardy and Lieb-Thirring inequalities for anyons*

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Abstract

We consider the many-particle quantum mechanics of *anyons*, i.e. identical particles in two space dimensions with a continuous statistics parameter $\alpha \in [0, 1]$ ranging from bosons ($\alpha = 0$) to fermions ($\alpha = 1$). We prove a (magnetic) Hardy inequality for anyons, which in the case that α is an odd numerator fraction implies a local exclusion principle for the kinetic energy of such anyons. From this result, and motivated by Dyson and Lenard's original approach to the stability of fermionic matter in three dimensions, we prove a Lieb-Thirring inequality for these types of anyons.

1 Introduction

The concept of identical particles and associated particle statistics lies at the foundations of quantum mechanics. It arises as a consequence of the non-observability of particle interchange and the fact that states in quantum mechanics are represented by rays in a complex Hilbert space, i.e. only determined up to a complex phase. A quantum mechanical state describing N distinguishable particles¹ moving in \mathbb{R}^d is represented by an N -particle *wavefunction*, i.e. a square-integrable complex-valued function $u \in L^2(\mathbb{R}^{dN})$ defined on N copies of \mathbb{R}^d , or equivalently, by an element of the tensor product space $\bigotimes^N L^2(\mathbb{R}^d)$ derived from the one-particle Hilbert space $L^2(\mathbb{R}^d)$. Upon restricting to *identical* particles, the freedom of choice of particle statistics stems from the fact that only the amplitude $|u(x)|$ of the wavefunction — describing (the square root of) the probability density for measuring the specific configuration $x = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N)$ of particle positions $\mathbf{x}_j \in \mathbb{R}^d$ — is observable, but not the exact phase $u(x)/|u(x)|$. Hence, since there should be no observable difference between the particle configuration

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¹We will for simplicity restrict to *scalar* non-relativistic particles, i.e. point particles without internal symmetries and spin.

$x = (\dots, \mathbf{x}_j, \dots, \mathbf{x}_k, \dots)$ and the one $x' = (\dots, \mathbf{x}_k, \dots, \mathbf{x}_j, \dots)$, with particles j and k interchanged, the amplitudes must be the same, but the phase may differ, as expressed by

$$u(x') = e^{i\alpha\pi} u(x), \quad (1)$$

with $\alpha \in [0, 2)$. In three and higher dimensions one finds that the only two possibilities for such a phase are $e^{i\alpha\pi} = \pm 1$, corresponding to *bosons* (such as photons) with the plus sign, and *fermions* (such as electrons) with the minus sign. It is in fact sufficient to consider the permutation group S_N acting on the N -particle Hilbert space $L^2(\mathbb{R}^{dN})$, and the wavefunctions describing identical bosons respectively fermions are then given by the completely symmetric resp. antisymmetric N -particle wavefunctions, which in the latter case can be represented by the Hilbert space $\bigwedge^N L^2(\mathbb{R}^d)$.

However, in two dimensions the statistics parameter α can be taken to be *any* real number in the interval $[0, 2)$ (or $(-1, 1]$, by periodicity). Again, bosons correspond to $\alpha = 0$ and fermions to $\alpha = 1$, while for a general choice of α the corresponding particles are simply called *anyons* (or, historically, just “particles obeying *intermediate* or *fractional statistics*” [9, 5, 20]). This discrepancy between two and higher spatial dimensions is directly related to the fact that a punctured plane is not simply connected, while $\mathbb{R}^d \setminus \{0\}$ is, for $d \geq 3$. Hence, for particles confined to the plane one has to consider *continuous* interchanges of particles, forming a loop in the configuration space (together with the possibility of enclosing other particles in that loop), and the permutation group symmetry is replaced by the *braid group* B_N , whose one-dimensional unitary representations determine the choice of statistics for identical anyons through the phase $e^{i\alpha\pi}$. For a convenient and rigorous treatment of anyons, one can model them as identical bosons or fermions in the plane, but with a magnetic interaction of Aharonov-Bohm type between each pair of particles, giving rise to the correct statistics phase as the particles encircle each other. The quantum mechanical momenta of the particles, which for bosons and fermions are simply given by the gradients $\nabla_j u$ w.r.t. the particle positions \mathbf{x}_j , will then be replaced by covariant (magnetic) derivatives $D_j u$ (see (12) below).

Fermions in any dimension are special, since they satisfy the so-called *Pauli exclusion principle*. Namely, because of the antisymmetry of the wavefunction, no two particles can occupy the exact same state, expressed simply with the help of the wedge product as $u_0 \wedge u_0 = 0$ for any one-particle state $u_0 \in L^2(\mathbb{R}^d)$. An important and non-trivial consequence of this is the celebrated *Lieb-Thirring inequality*, which given a scalar potential V on \mathbb{R}^d can

be summarized as

$$\begin{aligned} \sum_{j=1}^N \int_{\mathbb{R}^{dN}} (|\nabla_j u|^2 + V(\mathbf{x}_j)|u|^2) dx \\ \geq - \sum_{k=0}^{N-1} |\lambda_k(h)| \geq -C_d^{LT} \int_{\mathbb{R}^d} |V_-(\mathbf{x})|^{1+\frac{d}{2}} d\mathbf{x}, \end{aligned} \quad (2)$$

where $\lambda_k(h)$ denote the negative eigenvalues (ordered with decreasing magnitude) of the one-particle Schrödinger operator $h := -\Delta_{\mathbb{R}^d} + V(\mathbf{x})$ acting in $L^2(\mathbb{R}^d)$. The first inequality just expresses the fact that, because of the Pauli principle for fermions, the lowest possible energy (l.h.s. of (2)) is obtained when the particles assume the states corresponding to the lowest N eigenvalues $\lambda_k(h)$, i.e. when $u = \bigwedge_{k=0}^{N-1} u_k$ is an antisymmetrized product of those one-particle eigenfunctions $u_k(h)$. The second inequality holds uniformly in N and concerns the trace over the negative spectrum of the one-particle operator h . Ever since the first proof of (2) in 1975 by Lieb and Thirring [12], who used this result for a simplified proof of stability of fermionic matter (see also [11]), there has been a lot of activity in the mathematical community aiming to generalize this type of spectral estimate for one-particle operators in various directions. Furthermore, the Lieb-Thirring inequality (2) (disregarding the intermediate sum over eigenvalues) is equivalent to the *kinetic energy inequality*

$$T := \sum_{j=1}^N \int_{\mathbb{R}^{dN}} |\nabla_j u|^2 dx \geq C_d^K \int_{\mathbb{R}^d} \rho(\mathbf{x})^{1+\frac{2}{d}} d\mathbf{x}, \quad (3)$$

which can be interpreted as a strong form of the uncertainty principle for fermions, because of the way it bounds the total kinetic energy T in terms of the one-particle density

$$\rho(\mathbf{x}) := \sum_{j=1}^N \int_{\mathbb{R}^{d(N-1)}} |u(\mathbf{x}_1, \dots, \mathbf{x}_j = \mathbf{x}, \dots, \mathbf{x}_N)|^2 \prod_{k \neq j} d\mathbf{x}_k \quad (4)$$

of the wavefunction (always assumed to be normalized to $\int_{\mathbb{R}^{dN}} |u|^2 dx = 1$).

Bosons, on the other hand, do not satisfy any exclusion principle. They can all be put in the same state, e.g. $u = u_0 \otimes \dots \otimes u_0$, and hence cannot be expected to satisfy the inequalities (2) or (3). In fact, the best we can do is to treat them as N copies of a single particle satisfying (2), and hence the above inequalities hold only in the weaker form

$$\sum_{j=1}^N \int_{\mathbb{R}^{dN}} (|\nabla_j u|^2 + V(\mathbf{x}_j)|u|^2) dx \geq -C_d^{LT} N \int_{\mathbb{R}^d} |V_-(\mathbf{x})|^{1+\frac{d}{2}} d\mathbf{x},$$

resp. (cp. Appendix B)

$$\sum_{j=1}^N \int_{\mathbb{R}^{dN}} |\nabla_j u|^2 dx \geq \frac{C_d^K}{N^{2/d}} \int_{\mathbb{R}^d} \rho(\mathbf{x})^{1+\frac{2}{d}} d\mathbf{x},$$

which now only encodes the uncertainty principle, without the extra gain in (3) due to statistics. These bosonic inequalities become trivial as $N \rightarrow \infty$.

In contrast, not much has been known about the spectral and statistical properties of many anyons for $0 < |\alpha| < 1$. Not even the ground state energy of an otherwise non-interacting gas of anyons has been rigorously estimated². The main difficulty lies in the fact that many-anyon wavefunctions cannot be simply related to one-particle wavefunctions in the same way as for bosons and fermions. As will be seen explicitly below, the Hamiltonian operator $H_0 = \sum_{j=1}^N D_j^2$ describing the kinetic energy of N free anyons is not just a free Laplacian acting on totally symmetric or antisymmetric wavefunctions, but involves long-range magnetic interactions between all the particles. In particular, we cannot reduce our study to the relatively simple case of a one-particle Schrödinger operator h . The aim of the present paper is to address this gap in knowledge concerning intermediate anyon statistics, as well as the current lack of techniques to study the spectral theory of such quantum mechanical systems.

Our first main result concerns a magnetic many-particle Hardy inequality for N anyons, which when considered on the full two-dimensional plane \mathbb{R}^2 reads:

$$\sum_{j=1}^N \int_{\mathbb{R}^{2N}} |D_j u|^2 dx \geq \frac{4C_{\alpha,N}^2}{N} \sum_{i < j} \int_{\mathbb{R}^{2N}} \frac{|u|^2}{|\mathbf{x}_i - \mathbf{x}_j|^2} dx, \quad (5)$$

with the statistics-dependent constant

$$C_{\alpha,N} := \min_{p=0,1,\dots,N-2} \min_{q \in \mathbb{Z}} |(2p+1)\alpha - 2q|. \quad (6)$$

A stronger form of (5), given as Theorem 4 below, and valid for any convex subdomain $\Omega \subseteq \mathbb{R}^2$, is shown to produce a local form of *Pauli's exclusion principle for anyons* (given as Lemma 7 below), whose strength depends on the large- N behavior of the constant $C_{\alpha,N}$. Although $C_{\alpha,N}$ is clearly non-zero for all N whenever α is irrational, we find that $\inf_{N \in \mathbb{N}} C_{\alpha,N} = 0$, unless $\alpha = \frac{\mu}{\nu}$ with μ and ν relatively prime integers and μ odd, in which case we shall see that $\inf_{N \in \mathbb{N}} C_{\alpha,N} = \frac{1}{\nu}$.

For such *odd numerator fractions* $\alpha = \frac{\mu}{\nu}$, the energy given by the local Pauli exclusion principle for arbitrary numbers of particles is of a similar form as for fermions, but with an extra factor $\frac{1}{\nu^2}$ (depending wildly on the

² Note that this is trivial in the case of bosons, and a simple exercise in the case of fermions. The problem for anyons has been attacked by many authors through various approximations (see the references in the books and reviews [4, 7, 10, 16, 21]).

statistics parameter). Our second main result is that this local bound is sufficient to produce a Lieb-Thirring inequality for this type of anyons. In our approach we have been inspired by Dyson and Lenard's original proof of the stability of fermionic matter in three dimensions [2] (from 1967, before the advent of the Lieb-Thirring inequality), in which the only place where the Pauli principle came in was through such a local bound for the energy.

Theorem 1. *For a normalized wavefunction u of N anyons on \mathbb{R}^2 , with odd-fractional statistics parameter $\alpha = \frac{\mu}{\nu}$ (i.e. a reduced fraction with μ odd), we have the kinetic energy inequality*

$$T := \sum_{j=1}^N \int_{\mathbb{R}^{2N}} |D_j u|^2 dx \geq C_K \frac{1}{\nu^2} \int_{\mathbb{R}^2} \rho(\mathbf{x})^2 d\mathbf{x}, \quad (7)$$

where ρ is the corresponding one-particle density (4), and hence the Lieb-Thirring inequality

$$\sum_{j=1}^N \int_{\mathbb{R}^{2N}} (|D_j u|^2 + V(\mathbf{x}_j)|u|^2) dx \geq -C_{LT} \nu^2 \int_{\mathbb{R}^2} |V_-(\mathbf{x})|^2 d\mathbf{x}, \quad (8)$$

for any real-valued potential V on \mathbb{R}^2 . Here C_K and C_{LT} are universal positive constants that can be given explicitly.

In particular, this implies that the total kinetic energy T per unit area for a non-interacting (apart from the statistical interaction) gas of anyons with odd-fractional statistics $\alpha = \frac{\mu}{\nu}$, confined to an area A , is bounded below by

$$\frac{T}{A} \geq C_K \frac{\bar{\rho}^2}{\nu^2},$$

where $\bar{\rho} := N/A$ is the average density of the gas. Other implications of this Lieb-Thirring inequality, such as for interacting anyon gases, as well as the sharpness of this result concerning even-fractional and irrational statistics and its physical interpretation, will be discussed in a forthcoming paper [14].

The structure of this paper is as follows. In Section 2 we fix the notation and briefly recall the general theory of particle statistics. Our fundamental many-particle Hardy inequality for anyons is proven in Section 3 based on a pairwise relative parameterization of the configuration space, combined with a local magnetic Hardy inequality which takes into account the underlying symmetry between the particles. Section 4 concerns the local gain in energy following from these Hardy inequalities, which we refer to as a local Pauli exclusion principle for anyons due to its direct similarity with the corresponding local gain in energy for fermions. In Section 5 we use this local gain to prove the Lieb-Thirring inequality for anyons. In the appendices we have placed some suggestions for improvements of the local energy, as well

as a proof of a Lieb-Thirring inequality on cubes with Neumann boundary conditions.

We emphasize that, although a Lieb-Thirring-type inequality for anyons could perhaps have been anticipated based on physical grounds (at least for some values of the statistics parameter α), this is from a purely mathematical perspective a highly non-trivial extension of the usual Lieb-Thirring inequality (2) since the relevant operator

$$H = \sum_{j=1}^N (D_j(x)^2 + V(\mathbf{x}_j)),$$

with $D_j(x)$ being the differential operators given explicitly in (12) below, is now a strongly interacting magnetic many-particle Hamiltonian.

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2 Preliminaries

2.1 Particle statistics in two dimensions

In this subsection, which is *not* a prerequisite for understanding the rest of the paper, we give a very brief recap of the general theory of particle statistics in $d \geq 2$ dimensions. For more details we refer to the original reference [9], the review articles [4, 16], and the books [7, 10, 21] on anyons.

The classical configuration space of N identical particles in \mathbb{R}^d is formally given by

$$X_d^N := (\mathbb{R}^{dN} \setminus \mathbb{D}) / S_N,$$

where we have excluded all coincidences of the particles, i.e. the diagonals

$$\mathbb{D} := \{x \in \mathbb{R}^{dN} : \mathbf{x}_j = \mathbf{x}_k \text{ for some } j \neq k\},$$

and the symmetric group S_N acts on the N copies of \mathbb{R}^d in the obvious way. (The center-of-mass coordinate $\mathbf{X} := \frac{1}{N} \sum_j \mathbf{x}_j$ can be trivially factored out, $X_d^N \cong \mathbb{R}^d \times X_{d,\text{rel}}^N$, leaving the *relative* configuration space $X_{d,\text{rel}}^N = (\{x \in \mathbb{R}^{dN} : \mathbf{X} = 0\} \setminus \mathbb{D}) / S_N$.) For $d \geq 3$ the fundamental group of X_d^N is $\pi_1(X_d^N) = S_N$, whereas for $d = 2$ it is the braid group on N strands, $\pi_1(X_2^N) = B_N$. Wavefunctions of N identical particles are defined as square-integrable complex-valued functions on X_d^N with appropriate gluing conditions (recall the physical requirement (1)). Hence, these can be viewed as sections of a complex line bundle over X_d^N . There are natural flat connections on such line bundles, taking the trivial connection locally on \mathbb{R}^{dN} (note that the parallel transports could be *globally* non-trivial as there

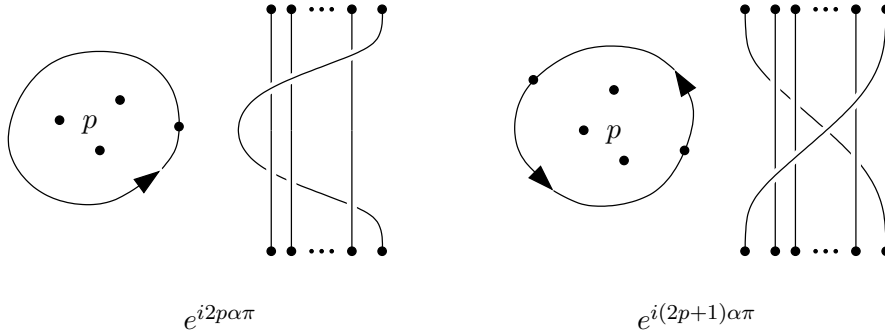


Figure 1: One- resp. two-particle interchange loops with corresponding braid diagrams (where we can think of time as running upwards) and phases.

are non-trivial transition functions between local regions), and every such connection defines a unitary one-dimensional representation of the fundamental group $\pi_1(X_d^N)$. For $d \geq 3$ there are only two such representations, the trivial one corresponding to bosons, and the sign on S_N corresponding to fermions. For $d = 2$ the unitary one-dimensional representations of the braid group are parameterized by a real number $\alpha \in [0, 2)$, where every generator in B_N corresponding to a counter-clockwise interchange of two neighbouring strands is represented by the phase $e^{i\alpha\pi}$. In particular, this will imply that

$$(\mathcal{P}_{(p)}u)(x) = e^{i(2p+1)\alpha\pi}u(x), \quad (9)$$

where $\mathcal{P}_{(p)}$ denotes the action of parallel transport along a closed loop in X_2^N corresponding to continuous counter-clockwise interchange of two particles \mathbf{x}_j and \mathbf{x}_k , with the interchange loop enclosing precisely $0 \leq p \leq N - 2$ other particles $\mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_p}$. On the other hand, if a single particle \mathbf{x}_j is taken along a simple loop which encloses p other particles, then a phase factor $e^{i2p\alpha\pi}$ will be picked up. See Figure 1 for these examples, and for a glimpse of how the exact phases arise through braid diagrams, with each elementary counter-clockwise braid contributing a phase $e^{i\alpha\pi}$. If $\alpha = 0$ or $\alpha = 1$ the representations depend only on the permutations of the particles and not on the braid, and we are back to the case of bosons and fermions.

We shall denote the Hilbert space of N -particle wavefunctions with a given statistics parameter α by \mathcal{H}_α^N . In the case of bosons or fermions we may identify such sections of line bundles with complex-valued functions on \mathbb{R}^{dN} that are either totally symmetric or totally antisymmetric, and hence we obtain the usual spaces of N -boson, resp. N -fermion wavefunctions.

In general, the line bundles corresponding to different α are topologically equivalent, but not geometrically equivalent. The map (singular gauge

transformation)

$$u(x) \mapsto \prod_{k < l} e^{i(\alpha - \alpha_0)\phi_{kl}} u(x), \quad \phi_{kl} := \arg \frac{\mathbf{x}_k - \mathbf{x}_l}{|\mathbf{x}_k - \mathbf{x}_l|}, \quad (10)$$

(where we pick an arbitrary real axis to identify \mathbb{R}^2 with \mathbb{C}) maps the line bundle corresponding to α_0 to the line bundle corresponding to α . The natural flat connection on the α_0 -bundle will then not be mapped to the natural flat connection on the α -bundle, but gives rise to the non-trivial gauge potential

$$\mathbf{A}_j := -i \left(\prod_{k < l} e^{i(\alpha - \alpha_0)\phi_{kl}} \right)^{-1} \nabla_{\mathbf{x}_j} \left(\prod_{k < l} e^{i(\alpha - \alpha_0)\phi_{kl}} \right). \quad (11)$$

We can then think of the α_0 -bundle as providing a reference statistic, and we will choose α_0 to be either 0 or 1 in order to model general statistics α in terms of bosonic or fermionic wavefunctions. In this way we can model N -anyon wavefunctions as either totally symmetric or antisymmetric N -particle wavefunctions $u \in \mathcal{H}_{\alpha_0}^N \subseteq L^2(\mathbb{R}^{2N})$ with covariant derivatives

$$D_j = -i\nabla_j + \mathbf{A}_j.$$

The advantage of taking this viewpoint is that we then can work exclusively on the redundant but uncomplicated configuration space \mathbb{R}^{2N} . The disadvantage, of course, is that we have to deal with a long-range magnetic interaction potential \mathbf{A}_j . In the physics literature the former viewpoint is often referred to as the *anyon gauge*, while the latter is called the *magnetic gauge*. Although we will mostly stick to the magnetic gauge (with $\alpha_0 := 0$), it is for the purpose of intuition very useful to have both pictures in mind.

2.2 Notation

In order to make precise our notation, we will denote by $\mathcal{D}_{\alpha_0, \alpha}^N$ the space of *finite kinetic energy* wavefunctions of N anyons with reference statistic $\alpha_0 \in \{0, 1\}$ and actual statistics parameter α , i.e. the totally symmetric/antisymmetric functions $u \in \mathcal{H}_{\alpha_0}^N \subseteq L^2(\mathbb{R}^{2N})$, where each particle is interacting with all the other through Aharonov-Bohm magnetic potentials of strength $\alpha - \alpha_0$, in the following precise sense. The covariant derivative acting w.r.t. particle \mathbf{x}_j (following from (10)–(11)) is

$$D_j := -i\nabla_j + \mathbf{A}_j(\mathbf{x}_j) := -i\nabla_{\mathbf{x}_j} + (\alpha - \alpha_0) \sum_{k \neq j} (\mathbf{x}_j - \mathbf{x}_k)^{-1} I, \quad (12)$$

where $\mathbf{a}^{-1} := \mathbf{a}/|\mathbf{a}|^2$ and $\mathbf{a} \mapsto \mathbf{a}I$ denotes counter-clockwise rotation of the vector $\mathbf{a} \in \mathbb{R}^2$ by an angle $\pi/2$. We consider the semi-bounded quadratic

form

$$u \mapsto q(u) := \sum_{j=1}^N \int_{\mathbb{R}^{2N}} |D_j u|^2 dx = \sum_{j=1}^N \int_{\mathbb{R}^{2N}} \bar{u} D_j^2 u dx, \quad (13)$$

defined initially on $u \in C_0^\infty(\mathbb{R}^{2N} \setminus \mathbb{D}) \cap \mathcal{H}_{\alpha_0}^N$, i.e. the smooth totally symmetric/antisymmetric and square-integrable functions on \mathbb{R}^{2N} with support away from diagonals (this amounts to what is in the literature sometimes referred to as a *hard-core condition* on anyons). The space of finite kinetic energy N -anyon wavefunctions $\mathcal{D}_{\alpha_0, \alpha}^N$ is then defined as the domain of the closure of this quadratic form q on $\mathcal{H}_{\alpha_0}^N$. Note that there is also an associated smaller space $\tilde{\mathcal{D}}_{\alpha_0, \alpha}^N$ defined as the domain of the Friedrichs extension, i.e. of the self-adjoint operator $H_0 = \sum_j D_j^2$ on $\mathcal{H}_{\alpha_0}^N \subseteq L^2(\mathbb{R}^{2N})$ associated to the closure of the quadratic form (13), and that for $\alpha_0 = \alpha = 0$,

$$\tilde{\mathcal{D}}_{0,0}^N = H^2(\mathbb{R}^{2N}) \cap \mathcal{H}_0^N \subseteq H^1(\mathbb{R}^{2N}) \cap \mathcal{H}_0^N = \mathcal{D}_{0,0}^N,$$

where H^k denote the Sobolev spaces of k partial derivatives in L^2 . In our proofs we will always use the denseness of the smooth functions with compact support in these spaces, and pick such representatives without taking explicit limits.

In the following, open resp. closed balls of radius r at a point x will be denoted $B_r(x)$, resp. $\bar{B}_r(x)$, and the characteristic function of a set A is denoted χ_A . Given a real-valued function or expression f , we define the non-negative quantities $f_\pm := \max\{0, \pm f\}$.

3 Hardy inequalities for anyons

We can mention [6], where a many-particle Hardy inequality has been derived for anyons (see Theorem 2.7 in [6]), but which is unfortunately not sufficient for our purposes³. The weakness stems from the fact that, so far, only single-particle movements have been taken into account, which only captures some of the symmetries involved (cp. Figure 1). In order to arrive at something non-trivial, at a minimum for the special case $\alpha = 1$ of fermions, we need to consider a *relative* Hardy inequality (cp. e.g. Lemma 4.6 in [6] which one could view as a relative Hardy inequality for a pair

³ Note, for instance, that the corresponding Hardy constant in [6],

$$D_{N,\alpha} = \min_{l=1,2,\dots,N-1} \left(\frac{\min_{k \in \mathbb{Z}} |l\alpha - k|}{l} \right)^2,$$

is zero for $\alpha = 1$ (and any $\alpha \in \mathbb{Q}$ for N large enough), and that in any case $D_{N,\alpha} \lesssim N^{-2}$. Hardy inequalities for interactions of anyonic type have also been considered in [15], but these are of single-particle type and hence also do not take the underlying symmetry between particles into account, as well as have an unclear dependence of the corresponding constants on the positions of the Aharonov-Bohm fluxes.

of fermions in any dimension). Also crucial for our approach is the following extension of a class of well-known two-dimensional magnetic Hardy inequalities (see [8, 1, 15]), where we take the underlying symmetry of the wavefunction and gauge potential into account.

Lemma 2 (Magnetic Hardy inequality with symmetry). *Let $\Omega = B_{R_2}(0) \setminus \bar{B}_{R_1}(0)$, $R_2 > R_1 \geq 0$, be an annular domain in \mathbb{R}^2 , and let there be a magnetic flux Φ inside $\bar{B}_{R_1}(0)$, determined on Ω by a vector potential $\mathbf{a} : \Omega \rightarrow \mathbb{R}^2$, s.t. $\nabla \wedge \mathbf{a} = 0$ on Ω and $\int_{\Gamma} \mathbf{a} \cdot d\mathbf{r} = \Phi$ for any simple loop Γ in Ω enclosing $\bar{B}_{R_1}(0)$. Furthermore, assume that \mathbf{a} is antipodal-antisymmetric, i.e. $\mathbf{a}(-\mathbf{r}) = -\mathbf{a}(\mathbf{r})$ for all $\mathbf{r} \in \Omega$, and let $v \in C^\infty(\Omega)$ be a function on Ω with antipodal symmetry, $v(-\mathbf{r}) = v(\mathbf{r})$ for all $\mathbf{r} \in \Omega$. Then*

$$\int_{\Omega} |D_{\mathbf{r}} v|^2 d\mathbf{r} \geq \min_{k \in \mathbb{Z}} \left| \frac{\Phi}{2\pi} - 2k \right|^2 \int_{\Omega} \frac{|v|^2}{|\mathbf{r}|^2} d\mathbf{r}, \quad (14)$$

where $D_{\mathbf{r}} := -i\nabla_{\mathbf{r}} + \mathbf{a}(\mathbf{r})$.

Alternatively, if v is antipodal-antisymmetric, $v(-\mathbf{r}) = -v(\mathbf{r})$ for all $\mathbf{r} \in \Omega$, then

$$\int_{\Omega} |D_{\mathbf{r}} v|^2 d\mathbf{r} \geq \min_{k \in \mathbb{Z}} \left| \frac{\Phi}{2\pi} - (2k+1) \right|^2 \int_{\Omega} \frac{|v|^2}{|\mathbf{r}|^2} d\mathbf{r}. \quad (15)$$

Proof. There exists a gauge transformation $v \mapsto \tilde{v} = e^{i\chi} v$ such that $|D_{\mathbf{r}} v|^2 = |(-i\nabla_{\mathbf{r}} + \tilde{\mathbf{a}})\tilde{v}|^2$, where $\mathbf{a} \mapsto \tilde{\mathbf{a}}(\mathbf{r}) := \frac{\Phi}{2\pi} \mathbf{r}^{-1} I$ on Ω . Note that $\chi(\mathbf{r})$, being the integral of a difference of two gauge potentials $\mathbf{a}(\mathbf{r})$ and $\tilde{\mathbf{a}}(\mathbf{r})$, both antisymmetric w.r.t. $\mathbf{r} \mapsto -\mathbf{r}$, must be symmetric under this antipodal map. Hence, if v is antipodal-(anti)symmetric, then so is \tilde{v} .

Now, we write the gauge-transformed l.h.s. of (14) in terms of polar coordinates (r, φ) ,

$$\int_{\Omega} |D_{\mathbf{r}} v|^2 d\mathbf{r} = \int_0^{2\pi} \int_{R_1}^{R_2} |\partial_r \tilde{v}|^2 r dr d\varphi + \int_0^{2\pi} \int_{R_1}^{R_2} \frac{1}{r^2} \left| \left(-i\partial_{\varphi} + \frac{\Phi}{2\pi} \right) \tilde{v} \right|^2 r dr d\varphi.$$

Considering only the last term involving ∂_{φ} , and Fourier expanding $\tilde{v}(\mathbf{r})$ on Ω ,

$$\tilde{v}(r, \varphi) = \tilde{v}(\mathbf{r} = r\mathbf{e}_1 e^{i\varphi}) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \tilde{v}_k(r) e^{i2k\varphi}$$

(note that we have here used the fact that $\tilde{v}(r, \varphi + \pi) = \tilde{v}(r, \varphi)$ for all φ), we find

$$\int_{R_1}^{R_2} \sum_{k \in \mathbb{Z}} \left| 2k + \frac{\Phi}{2\pi} \right|^2 \frac{|\tilde{v}_k|^2}{r^2} r dr \geq \min_{k \in \mathbb{Z}} \left| \frac{\Phi}{2\pi} - 2k \right|^2 \int_{\Omega} \frac{|v|^2}{|\mathbf{r}|^2} d\mathbf{r},$$

and hence arrive at the inequality (14). For the case of antipodal-antisymmetric v , we can Fourier expand \tilde{v} in odd powers of $e^{ik\varphi}$ and arrive at (15). \square

Now, let us for simplicity first apply this to the case of only two anyons and prove a relative Hardy inequality for this system.

Lemma 3 (Relative two-anyon Hardy). *Let Ω be an open convex set in \mathbb{R}^2 and let $u \in \mathcal{D}_{0,\alpha}^2$ be a two-anyon wavefunction. Then*

$$\int_{\Omega \circ \Omega} (|D_1 u|^2 + |D_2 u|^2) d\mathbf{x}_1 d\mathbf{x}_2 \geq 2 \min_{k \in \mathbb{Z}} |\alpha - 2k|^2 \int_{\Omega \circ \Omega} \frac{|u|^2}{|\mathbf{x}_1 - \mathbf{x}_2|^2} d\mathbf{x}_1 d\mathbf{x}_2, \quad (16)$$

where

$$\Omega \circ \Omega := \{(\mathbf{x}_1, \mathbf{x}_2) \in \Omega^2 : \tfrac{1}{2}|\mathbf{x}_1 - \mathbf{x}_2| < \text{dist}(\tfrac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2), \Omega^c)\} \quad (17)$$

(in particular, $\Omega \circ \Omega \subsetneq \Omega^2$, unless $\Omega = \mathbb{R}^2$).

Proof. Let us introduce the center-of-mass $\mathbf{R} := \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$ and the relative coordinate $\mathbf{r} := \frac{1}{2}(\mathbf{x}_1 - \mathbf{x}_2)$. Furthermore, let $v(\mathbf{R}; \mathbf{r}) := u(\mathbf{R} + \mathbf{r}, \mathbf{R} - \mathbf{r})$, and observe that the bosonic symmetry of $u \in \mathcal{H}_0^2$ implies $v(\mathbf{R}; -\mathbf{r}) = v(\mathbf{R}; \mathbf{r})$ for all $\mathbf{R} \in \Omega$ and $\mathbf{r} \in \mathbb{R}^2$ s.t. $0 < |\mathbf{r}| < \text{dist}(\mathbf{R}, \Omega^c) =: \delta(\mathbf{R})$ (possibly infinite). Then, by $\nabla_{\mathbf{R}} = \nabla_1 + \nabla_2$, $\nabla_{\mathbf{r}} = \nabla_1 - \nabla_2$, the l.h.s. of (16) equals

$$\begin{aligned} & \int_{\Omega} \int_{B_{\delta(\mathbf{R})}(0)} \left(\left| \left(-\frac{i}{2} \nabla_{\mathbf{R}} - \frac{i}{2} \nabla_{\mathbf{r}} + \frac{\alpha}{2} \mathbf{r}^{-1} I \right) v \right|^2 \right. \\ & \quad \left. + \left| \left(-\frac{i}{2} \nabla_{\mathbf{R}} + \frac{i}{2} \nabla_{\mathbf{r}} - \frac{\alpha}{2} \mathbf{r}^{-1} I \right) v \right|^2 \right) 2d\mathbf{r} d\mathbf{R} \\ & = \int_{\Omega \circ \Omega} |\nabla_{\mathbf{R}} v|^2 d\mathbf{r} d\mathbf{R} + \int_{\Omega} \int_{B_{\delta(\mathbf{R})}(0)} |(-i \nabla_{\mathbf{r}} + \alpha \mathbf{r}^{-1} I) v|^2 d\mathbf{r} d\mathbf{R}. \end{aligned}$$

For the last integral w.r.t. \mathbf{r} we can then apply Lemma 2 (where no gauge transformation is necessary in this case), and thus find

$$\int_{\Omega \circ \Omega} (|D_1 u|^2 + |D_2 u|^2) d\mathbf{x}_1 d\mathbf{x}_2 \geq \min_{k \in \mathbb{Z}} |\alpha - 2k|^2 \int_{\Omega} \int_{B_{\delta(\mathbf{R})}(0)} \frac{|u|^2}{|\mathbf{r}|^2} d\mathbf{r} d\mathbf{R},$$

which implies (16). \square

An equivalent way to think about this result in terms of the anyon gauge, i.e. the space \mathcal{H}_{α}^2 , is that we can make a symmetric interchange of the two anyons by rotating them around their common center-of-mass. After half a turn — already then completing a loop in the relative configuration space $X_{2,\text{rel}}^2 = (\mathbb{R}^2 \setminus \{0\}) /_{\mathbf{r} \leftrightarrow -\mathbf{r}}$ — we can compare the function values, and the condition (9) on parallel transport tells us that we should pick up a phase $e^{i\alpha\pi}$. We are therefore given a function $v(\mathbf{r})$ on, say, the upper half-plane with a periodic boundary condition $v(-r\mathbf{e}_1) = e^{i\alpha\pi} v(r\mathbf{e}_1)$, and hence the Hardy inequality follows.

This is straightforwardly extended using Lemma 2 to cases when there are also fluxes inside the interchange loop. Intuitively, as we encircle p anyons with a symmetric two-anyon interchange loop we pick up a phase $e^{i(2p+1)\alpha\pi}$ and hence a corresponding Hardy inequality. More precisely, this results in the following relative many-particle Hardy inequality for anyons.

Theorem 4 (Many-anyon Hardy). *Let Ω be an open convex set in \mathbb{R}^2 and let $u \in \mathcal{D}_{0,\alpha}^N$ be an N -anyon wavefunction. Then*

$$\int_{\Omega^N} \sum_{j=1}^N |D_j u|^2 dx \geq \frac{4C_{\alpha,N}^2}{N} \int_{\Omega^N} \sum_{i < j} \frac{|u|^2}{|\mathbf{x}_i - \mathbf{x}_j|^2} \chi_{\Omega \circ \Omega}(\mathbf{x}_i, \mathbf{x}_j) dx, \quad (18)$$

with $\Omega \circ \Omega$ defined in (17), and

$$C_{\alpha,N} := \min_{p=0,1,\dots,N-2} \min_{q \in \mathbb{Z}} |(2p+1)\alpha - 2q|. \quad (19)$$

Proof. We use that, for any $z = (\mathbf{z}_j) \in \mathbb{C}^{dN}$,

$$\sum_{j=1}^N |\mathbf{z}_j|^2 = \frac{1}{N} \sum_{1 \leq j < k \leq N} |\mathbf{z}_j - \mathbf{z}_k|^2 + \frac{1}{N} \left| \sum_{j=1}^N \mathbf{z}_j \right|^2. \quad (20)$$

Applying this identity with $\mathbf{z}_j := D_j u(x)$ and dropping the second term, we see that the l.h.s. of (18) is bounded below by a sum of $\binom{N}{2}$ integrals of the form

$$\int_{\Omega^{N-2}} \int_{(\mathbf{x}_j, \mathbf{x}_k) \in \Omega^2} |(D_j - D_k)u|^2 d\mathbf{x}_j d\mathbf{x}_k \prod_{\substack{l=1,\dots,N \\ l \neq j,k}} d\mathbf{x}_l. \quad (21)$$

Hence, we consider for each fixed choice of the $N-2$ variables (\mathbf{x}_l) the remaining configurations of the pair $(\mathbf{x}_j, \mathbf{x}_k)$ on $\Omega \circ \Omega \subseteq \Omega^2$, which we again parameterize by a center-of-mass $\mathbf{R} := \frac{1}{2}(\mathbf{x}_j + \mathbf{x}_k)$ and relative coordinate $\mathbf{r} := \frac{1}{2}(\mathbf{x}_j - \mathbf{x}_k)$. For each $\mathbf{R} \in \Omega$ we can then split up the final parameterization of $\mathbf{r} \in B_{\delta(\mathbf{R})}(0) \setminus \{0\}$ into annuli, the first annulus extending from $r = \delta_0 := 0$ to $r = \delta_1$, defined to be the distance from \mathbf{R} (where there could possibly be one particle \mathbf{x}_l) to the next closest particle $\mathbf{x}_{l'}$ (or possibly several particles situated on the same distance from \mathbf{R}). The next annulus then extends from $r = \delta_1$ to the next greater distance δ_2 from \mathbf{R} to any particles, and so on, until we reach the boundary of the domain (or infinity) at $r = \delta_M := \delta(\mathbf{R})$. Now, on each annulus $A_m := B_{\delta_m} \setminus \bar{B}_{\delta_{m-1}}$, $m = 1, \dots, M$, we have

$$(D_j - D_k)u = (-i\nabla_{\mathbf{r}} + \alpha \mathbf{r}^{-1} I + \mathbf{a}(\mathbf{R}; \mathbf{r})) v,$$

where $v(\mathbf{R}; \mathbf{r}) := u(\mathbf{R} + \mathbf{r}, \mathbf{R} - \mathbf{r})$ is antipodal-symmetric in \mathbf{r} , while the gauge potential $\mathbf{a}(\mathbf{R}; \mathbf{r}) := \mathbf{A}(\mathbf{R} + \mathbf{r}) - \mathbf{A}(\mathbf{R} - \mathbf{r})$ is antipodal-antisymmetric, with $\mathbf{A}(\mathbf{x}) := \alpha \sum_{l \neq j,k} (\mathbf{x} - \mathbf{x}_l)^{-1} I$ being the magnetic potential at $\mathbf{x} \in$

$\mathbf{R} + A_m$ from all the other $N - 2$ particles \mathbf{x}_l . Note that $\nabla_{\mathbf{r}} \wedge \mathbf{a} = 0$ on A_m , and that $\int_{\Gamma} \mathbf{a} \cdot d\mathbf{r} = 4\pi\alpha p_m$ for any simple loop $\Gamma \subset A_m$ enclosing $\bar{B}_{\delta_{m-1}}$, where p_m is the number of particles \mathbf{x}_l inside $\bar{B}_{\delta_{m-1}}$. We can therefore apply Lemma 2, with the total flux Φ in the disk $\bar{B}_{\delta_{m-1}}$ given by $\Phi = 2\pi\alpha(1+2p_m)$. Hence,

$$\begin{aligned} \int_{A_m} |(D_j - D_k)u|^2 d\mathbf{r} &\geq 4 \min_{q \in \mathbb{Z}} |(2p_m + 1)\alpha - 2q|^2 \int_{A_m} \frac{|u|^2}{|\mathbf{x}_j - \mathbf{x}_k|^2} d\mathbf{r} \\ &\geq 4C_{\alpha,N}^2 \int_{A_m} \frac{|u|^2}{|\mathbf{x}_j - \mathbf{x}_k|^2} d\mathbf{r}, \end{aligned}$$

and proceeding similarly for all annuli A_m , all points \mathbf{R} , and all pairs (j, k) , we obtain the inequality (18). \square

Note that this theorem coincides with Lemma 3 for $N = 2$, and with Theorem 2.8 in [6] for fermions on \mathbb{R}^2 (since $C_{\alpha=1,N} = 1$ for all $N \geq 2$). We also note that $C_{\alpha=0,N} = 0$ is the optimal constant for bosons since (18) concerns the *Neumann* form, i.e. we could in this case (and if Ω has finite measure) take u to be constant so that the l.h.s. is identically zero. If one considers the *Dirichlet* form on the other hand, a non-trivial many-particle Hardy-type inequality for bosons or distinguishable particles in two dimensions (also with a constant $\sim N^{-1}$, although with logarithmic factors in the potentials) was derived in [13].

Concerning intermediate statistics, we have the following observation:

Proposition 5. *The infimum*

$$C_{\alpha} := \inf_{N \in \mathbb{N}} C_{\alpha,N} = \inf_{p,q \in \mathbb{Z}} |(2p+1)\alpha - 2q|$$

equals

$$C_{\alpha} = \begin{cases} \frac{1}{\nu}, & \text{if } \alpha = \frac{\mu}{\nu} \text{ with } \mu \in \mathbb{Z}, \nu \in \mathbb{N}_+ \text{ relatively prime and } \mu \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, C_{α} is strictly positive whenever α is an odd numerator fraction, but zero otherwise.

For an upper bound in the non-trivial cases we will use the following fact:

Lemma 6. *Given $a, b \in \mathbb{Z}$ coprime, where a is odd, there exist integers x, y such that x is odd, y is even, and $ax + by = 1$.*

Proof. Since a, b are coprime, we can by Euclid's algorithm find integers x, y s.t. $ax + by = 1$. We then use that $ax + by = a(x + kb) + b(y - ka)$ for any $k \in \mathbb{Z}$ and consider the different possibilities. If b is odd then, since a is

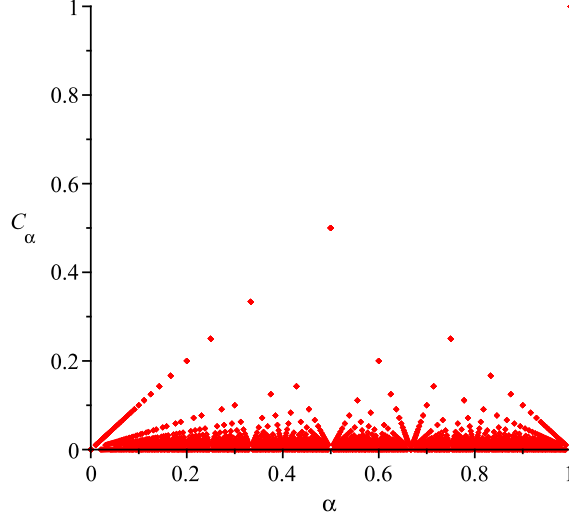


Figure 2: A sketch of the behavior of C_α as a function of α .

also odd, either x is odd and y even and we are done, or x is even and y is odd, but then we can choose k odd and so we are done. In the case that b is even, then either x is odd and y even and we are done, or both x and y are odd. In the latter case we can again choose k odd and we are done. \square

Proof of Proposition 5. Assume that $\alpha = \frac{\mu}{\nu}$, with $\mu \in \mathbb{Z}$, $\nu \in \mathbb{N}_+$ coprime, and consider first the case of odd numerators: $\mu = 2k + 1$, $k \in \mathbb{Z}$. Then

$$|(2p+1)\alpha - 2q| = \frac{1}{\nu} |(2p+1)(2k+1) - 2q\nu| \geq \frac{1}{\nu}, \quad \forall p, q \in \mathbb{Z},$$

i.e. $C_\alpha \geq \frac{1}{\nu}$, since the last absolute value is a difference between an odd and an even integer. Furthermore, Lemma 6 guarantees the existence of $p, q \in \mathbb{Z}$ such that this absolute value is equal to one, hence $C_\alpha = \frac{1}{\nu}$.

In the case of even numerator fractions, i.e. $\mu = 2k$, $k \in \mathbb{Z}$, we have $|(2p+1)\alpha - 2q| = \frac{2}{\nu} |(2p+1)k - \nu q|$. As ν necessarily is odd, we can choose $2p+1 = \nu$ and $q = k$, and hence $C_\alpha = 0$.

In the irrational case $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we can find an infinite sequence of $p, q \in \mathbb{Z}$ such that (see [19])

$$\left| \alpha - \frac{2q}{2p+1} \right| < \frac{1}{(2p+1)^2},$$

and hence $C_\alpha = 0$. \square

In Figure 2 we have plotted a sketch of the dependence on α of the large- N constant C_α . It should be emphasized that for small numbers of

particles (relative to a fixed α), the Hardy constant $C_{\alpha,N}$ is typically non-zero. Its graph can be obtained by cutting out wedges of slopes $2p+1$, $p = 0, 1, \dots, N-1$, at every rational point with denominator $2p+1$ and an even numerator. In particular, $C_{\alpha,N} > 0$ for all irrational α , and for even numerator fractions $\alpha = \frac{\mu}{\nu} \in (0, 1)$ the constant is strictly positive if and only if $N < \frac{\nu+3}{2}$. Let us also point out that we could just as well have chosen to work with the fermionic reference statistic, i.e. $\alpha_0 = 1$, for which the corresponding statistics-dependent constant is

$$\tilde{C}_{\beta,N} := \min_{p=0,1,\dots,N-2} \min_{q \in \mathbb{Z}} |(2p+1)\beta - (2q+1)|. \quad (22)$$

We have here denoted the strength of the statistical interaction by $\beta := \alpha - 1$, and the values of β for which this constant is bounded away from zero for all N are the fractions where either the numerator or denominator is even.

4 A local Pauli exclusion principle for the kinetic energy

Note that for odd-fractional statistics, i.e. for non-zero values of C_α , the total constant in the many-anyon Hardy inequality (18) still tends to zero like N^{-1} . Naively⁴, this would be insufficient for a non-trivial bound on the energy of the anyon gas in the thermodynamic limit, since the energy per area (L^2 , say, to which the gas is confined) due to Theorem 4 yields

$$\frac{T}{L^2} \geq \frac{1}{L^2} \cdot \frac{4C_\alpha^2}{N} \cdot \frac{\binom{N}{2}}{2L^2} \cdot \int |u|^2 dx = \text{const} \cdot \frac{N-1}{N^2} \cdot \bar{\rho}^2 \rightarrow 0,$$

as $N \rightarrow \infty$, with fixed density $\bar{\rho} := N/L^2$. However, Theorem 4 is stronger than that, and we can choose not to consider this global gain in energy due to statistics directly, but rather its *local* implications upon cutting the space up and employing Neumann boundary conditions. This local approach is in the spirit of Dyson and Lenard's original proof of the stability of matter [2]. By choosing the size of such local regions appropriately, the energy can then be lifted to a stronger bound on the full domain \mathbb{R}^2 (see (31) and (36) below).

Central for this approach is the following version of Lemma 5 in [2] for anyons. We refer to this as a local exclusion principle for the kinetic energy of anyons, since it implies that $n \geq 2$ anyons must have positive energy and therefore cannot all occupy the lowest zero energy state.

⁴Note that the sharp large- N behavior of this constant is not clear even in the fermionic case, cp. [6, 3]. We also remark that for $d \geq 3$ the corresponding sharp behavior is known [3], and does imply a non-trivial bound for the ground state energy of the fermion gas.

Lemma 7 (Local energy / Pauli principle). *Let $u \in \mathcal{D}_{0,\alpha}^n$ be a wavefunction of n anyons and let $\Omega \subseteq \mathbb{R}^2$ be either a disk or a square, with area $|\Omega|$. Then*

$$\int_{\Omega^n} \sum_{j=1}^n |D_j u|^2 dx \geq (n-1) \frac{c_\Omega C_{\alpha,n}^2}{|\Omega|} \int_{\Omega^n} |u|^2 dx, \quad (23)$$

where c_Ω is a constant which satisfies $c_\Omega \geq 0.477$ for the disk, and $c_\Omega \geq 0.358$ for the square.

Proof. Note that by rescaling, we can in the following assume that $\Omega = B_1(0)$ or $\Omega = [-1, 1]^2$. We shall first consider the case of the disk, and then point out what needs to be changed for the square.

Due to $\Omega \circ \Omega \subsetneq \Omega^2$, the bound given by the many-anyon Hardy inequality (18) is unfortunately not sufficient as it stands, and we need to modify the approach in the proof of Theorem 4 to take the whole two-particle domain Ω^2 into account. Instead of (20) we shall therefore use

$$\sum_{j=1}^n |z_j|^2 = \sum_{j < k} \left(\frac{1-\kappa}{n-1} (|z_j|^2 + |z_k|^2) + \frac{\kappa}{n} |z_j - z_k|^2 \right) + \frac{\kappa}{n} \left| \sum_j z_j \right|^2, \quad (24)$$

with $0 < \kappa < 1$. The last term is again thrown away while the middle one is employed as in the proof of Theorem 4 to produce a Hardy potential in terms of relative coordinates \mathbf{r} and \mathbf{R} . However, for the first two terms we instead use that $|D_j u|^2 \geq |\nabla_j |u||^2$ (diamagnetic inequality). Ignoring κ for a second, we are hence interested in the infimum of the ratio

$$\int_{\Omega^2} \left(|\nabla_{\mathbf{x}_1} u|^2 + |\nabla_{\mathbf{x}_2} u|^2 + \frac{C_{\alpha,n}^2}{|\mathbf{r}|^2} \chi_{B_\delta(\mathbf{R})(0)}(\mathbf{r}) |u|^2 \right) d\mathbf{x}_1 d\mathbf{x}_2 \Big/ \int_{\Omega^2} |u|^2 d\mathbf{x}_1 d\mathbf{x}_2 \quad (25)$$

over $u \in H^1(\Omega^2)$, which is certainly greater than the lowest eigenvalue of the Schrödinger operator ($c := C_{\alpha,n} \neq 0$ in the following)

$$H := -\Delta_{\Omega^2}^{\mathcal{N}} + f, \quad f(\mathbf{x}_1, \mathbf{x}_2) := c^2 g(|\mathbf{R}|, |\mathbf{r}|), \quad (26)$$

on Ω^2 defined with Neumann boundary conditions on $\partial\Omega^2$, with

$$g(R, r) := \begin{cases} \delta^{-2} (1 - \hat{R})^{-2}, & R \leq \hat{R}, \quad r \leq \delta(1 - R), \\ r^{-2}, & R \leq \hat{R}, \quad \delta(1 - R) < r < 1 - R, \\ 0, & R > \hat{R}, \end{cases}$$

for some fixed cut-off parameters $0 < \delta, \hat{R} < 1$, to be optimized over later.

Now, denoting by P the projection onto the constant function $u_0(x) := |\Omega^2|^{-\frac{1}{2}} = \pi^{-1}$, and $Q := 1 - P$, we have $(-\Delta_{\Omega^2}^{\mathcal{N}})P = 0$, and for the first non-zero Neumann eigenvalue $\lambda_1 = \lambda_1(-\Delta^{\mathcal{N}})$,

$$(-\Delta_{\Omega^2}^{\mathcal{N}})Q \geq \lambda_1(-\Delta_{\Omega^2}^{\mathcal{N}})Q = \lambda_1(-\Delta_{\Omega}^{\mathcal{N}})Q = \xi^2 Q,$$

where $\xi \approx 1.8412$ denotes the first zero of the derivative of the Bessel function J_1 . Furthermore, we have since

$$\begin{aligned} \langle u, (PfQ + QfP)u \rangle &= \langle f^{\frac{1}{2}}Pu, f^{\frac{1}{2}}Qu \rangle + \langle f^{\frac{1}{2}}Qu, f^{\frac{1}{2}}Pu \rangle \\ &\leq 2\|f^{\frac{1}{2}}Pu\| \|f^{\frac{1}{2}}Qu\| \leq \mu\|f^{\frac{1}{2}}Pu\|^2 + \frac{1}{\mu}\|f^{\frac{1}{2}}Qu\|^2 = \langle u, (\mu PfP + \mu^{-1}QfQ)u \rangle, \end{aligned}$$

for $u \in L^2(\Omega^2)$ and $\mu > 0$, that

$$f = (P + Q)f(P + Q) \geq (1 - \mu)PfP + (1 - \mu^{-1})QfQ.$$

These operators are estimated according to $\|QfQ\| \leq \|f\|_\infty = \frac{c^2}{\delta^2(1-\hat{R})^2}$, and

$$\begin{aligned} \|PfP\| &= \int_{\Omega^2} f u_0 d\mathbf{x}_1 d\mathbf{x}_2 = \frac{1}{\pi} \int_{\Omega^2} c^2 g(R, r) 2dr d\mathbf{R} \\ &= \frac{2(2\pi)^2 c^2}{\pi} \int_0^{\hat{R}} \left(\int_0^{\delta(1-R)} \delta^{-2}(1-\hat{R})^{-2} r dr + \int_{\delta(1-R)}^{1-R} \frac{1}{r} dr \right) R dR \\ &= 4\pi c^2 \left(\frac{1}{2} + \ln \delta^{-1} \right) \hat{R}^2, \end{aligned}$$

and hence

$$\begin{aligned} H &\geq (-\Delta_{\Omega_R}^{\mathcal{N}})P + (-\Delta_{\Omega_R}^{\mathcal{N}})Q + (1 - \mu)PfP + (1 - \mu^{-1})QfQ \\ &\geq (1 - \mu)2\pi c^2(1 + 2\ln \delta^{-1})\hat{R}^2 P + \left(\xi^2 - (\mu^{-1} - 1) \frac{c^2}{\delta^2(1 - \hat{R})^2} \right) Q. \end{aligned}$$

With the parameter κ from (24) reintroduced into (25) and (26), we bound

$$\begin{aligned} H &\geq 2\pi c^2(1 - \mu)(1 + 2\ln \delta^{-1})\hat{R}^2 \kappa P + \left(\xi^2(1 - \kappa) - \frac{(\mu^{-1} - 1)\kappa}{\delta^2(1 - \hat{R})^2} \right) Q \\ &\geq 0.304c^2, \end{aligned}$$

where the lower bound was found by numerical optimization, with $\mu = 0.851$, $\delta = 0.54899$, $\hat{R} = 0.54396$, $\kappa = 0.499$, $\xi^2 \geq 3.389$. Summing up, we have

$$\begin{aligned} &\sum_j \int_{\Omega^n} |D_j u|^2 dx \\ &\geq \frac{1}{n} \sum_{j < k} \int_{\Omega^{n-2}} \int_{\Omega^2} ((1 - \kappa)(|D_j u|^2 + |D_k u|^2) + \kappa f(\mathbf{x}_j, \mathbf{x}_k)) d\mathbf{x}_j d\mathbf{x}_k dx' \\ &\geq (n - 1) \frac{c_\Omega c^2}{|\Omega|} \int_{\Omega^n} |u|^2 dx, \quad (27) \end{aligned}$$

with $c_\Omega \geq 0.304 \cdot \pi/2 \geq 0.477$.

In the case that $\Omega = [-1, 1]^2 \supseteq B_1(0)$, we can use the same f and g as above (extended by zero outside $B_1(0)^2$), so that the bound on $\|QfQ\|$ is unchanged, $\|PfP\|$ is multiplied by the area ratio $\frac{\pi}{4}$, and $(-\Delta_{\Omega^2}^N)Q \geq \frac{\pi^2}{4}Q$. We find

$$\begin{aligned} H &\geq \frac{\pi^2 c^2}{2}(1 - \mu)(1 + 2 \ln \delta^{-1}) \hat{R}^2 \kappa P + \left(\frac{\pi^2}{4}(1 - \kappa) - \frac{(\mu^{-1} - 1)\kappa}{\delta^2(1 - \hat{R})^2} \right) Q \\ &\geq 0.179c^2, \end{aligned}$$

with $\mu = 0.8879$, $\delta = 0.5451$, $\hat{R} = 0.531$, $\kappa = 0.52$, and hence (27) holds with the constant $c_\Omega \geq 0.179 \cdot 4/2 = 0.358$. \square

Remark. More general convex domains can be treated in a similar way using [17]. Also note that the bound (23) for the disk holds with $c_\Omega = \pi\xi^2 \approx 10.65$ in the case that $\alpha = 1$ (compare with Lemma 5 in [2]). For possible ways of improving the constant c_Ω in the general case, see Appendix A.

5 A Lieb-Thirring inequality for anyons

Given a *normalized* N -anyon wavefunction $u \in \mathcal{H}_{\alpha_0}^N$ we will in the following denote by

$$\rho(\mathbf{x}) := \sum_{j=1}^N \int_{\mathbb{R}^{2(N-1)}} |u(\mathbf{x}_1, \dots, \mathbf{x}_{j-1}, \mathbf{x}, \mathbf{x}_{j+1}, \dots, \mathbf{x}_N)|^2 \prod_{k \neq j} d\mathbf{x}_k$$

the one-particle density, such that $\int_{\mathbb{R}^2} \rho(\mathbf{x}) d\mathbf{x} = N$. We start by reformulating Lemma 7 in terms of ρ .

Lemma 8 (Local Pauli principle). *Let $u \in \mathcal{D}_{0,\alpha}^N$ be an N -anyon wavefunction on \mathbb{R}^2 and $\Omega \subseteq \mathbb{R}^2$ a simply connected domain on which (23) holds for some constant c_Ω . Then*

$$T_\Omega := \sum_{j=1}^N \int_{\mathbb{R}^{2N}} |D_j u|^2 \chi_\Omega(\mathbf{x}_j) d\mathbf{x} \geq \frac{c_\Omega C_{\alpha,N}^2}{|\Omega|} \left(\int_\Omega \rho - 1 \right). \quad (28)$$

Proof. Using that

$$1 = \prod_{k=1}^N (\chi_\Omega(\mathbf{x}_k) + (1 - \chi_\Omega(\mathbf{x}_k))) = \sum_{A \subseteq \{1, \dots, N\}} \prod_{k \in A} \chi_\Omega(\mathbf{x}_k) \prod_{k \notin A} (1 - \chi_\Omega(\mathbf{x}_k)),$$

the l.h.s. of (28) is

$$\begin{aligned} T_\Omega &= \int_{\mathbb{R}^{2N}} \sum_{j=1}^N |D_j u|^2 \chi_\Omega(\mathbf{x}_j) \sum_{A \subseteq \{1, \dots, N\}} \prod_{k \in A} \chi_\Omega(\mathbf{x}_k) \prod_{k \notin A} (1 - \chi_\Omega(\mathbf{x}_k)) d\mathbf{x} \\ &= \sum_{A \subseteq \{1, \dots, N\}} \int_{\mathbb{R}^{2N}} \sum_{j \in A} |D_j u|^2 \prod_{k \in A} \chi_\Omega(\mathbf{x}_k) \prod_{k \notin A} (1 - \chi_\Omega(\mathbf{x}_k)) d\mathbf{x}. \end{aligned}$$

We now apply Lemma 7 to each term in the first summation above, which involves a partition A of the N particles into $n := |A|$ of them being inside the domain Ω , while the remaining $N - n$ residing outside, and therefore whose contributions to the magnetic potentials $\mathbf{A}_{j \in A}$ can be gauged away. Thus, we find

$$\begin{aligned} T_\Omega &\geq \sum_{A \subseteq \{1, \dots, N\}} \frac{c_\Omega C_{\alpha, |A|}^2}{|\Omega|} (|A| - 1)_+ \int_{\mathbf{x}_{k \notin A} \notin \Omega} \int_{\mathbf{x}_k \in A \in \Omega} |u|^2 \prod_{k \in A} d\mathbf{x}_k \prod_{k \notin A} d\mathbf{x}_k \\ &\geq \frac{c_\Omega C_{\alpha, N}^2}{|\Omega|} \int_{\mathbb{R}^{2N}} |u|^2 \sum_{A \subseteq \{1, \dots, N\}} (|A| - 1) \prod_{k \in A} \chi_\Omega(\mathbf{x}_k) \prod_{k \notin A} (1 - \chi_\Omega(\mathbf{x}_k)) dx. \end{aligned}$$

We then revert the above procedure using $\int |u|^2 = 1$ and

$$\begin{aligned} &\sum_{A \subseteq \{1, \dots, N\}} \underbrace{|A|}_{=\sum_{j \in A}} \prod_{k \in A} \chi_\Omega(\mathbf{x}_k) \prod_{k \notin A} (1 - \chi_\Omega(\mathbf{x}_k)) \\ &= \sum_{j=1}^N \sum_{A \subseteq \{1, \dots, N\}} \chi_\Omega(\mathbf{x}_j) \prod_{k \in A} \chi_\Omega(\mathbf{x}_k) \prod_{k \notin A} (1 - \chi_\Omega(\mathbf{x}_k)) = \sum_{j=1}^N \chi_\Omega(\mathbf{x}_j), \end{aligned}$$

which produces (28). \square

Lemma 9 (Local uncertainty principle / Neumann Lieb-Thirring inequality). *Let $u \in \mathcal{D}_{0, \alpha}^N$ be an N -anyon wavefunction on \mathbb{R}^2 , and Q a square with area $|Q|$. Then*

$$\sum_{j=1}^N \int_{\mathbb{R}^{2N}} |D_j u|^2 \chi_Q(\mathbf{x}_j) dx \geq C'_2 (\int_Q \rho)^{-1} \int_Q \left[\rho(\mathbf{x})^{\frac{1}{2}} - \left(\frac{\int_Q \rho}{|Q|} \right)^{\frac{1}{2}} \right]_+^4 d\mathbf{x}. \quad (29)$$

Proof. We use $|D_j u| \geq |\nabla_j u|$ and then apply the Neumann Lieb-Thirring inequality given in Theorem 14 in the appendix to the bosonic wavefunction $|u|$. \square

Remark. Note that the limit case $Q = \mathbb{R}^2$, $N = 1$, is just a special case of the Sobolev-type inequality $\|\nabla u\|_2^2 \geq C_p \|u\|_2^{-4/(p-2)} \|u\|_p^{2p/(p-2)}$ with $p = 4$.

Lemma 10. *For any domain $\Omega \subseteq \mathbb{R}^2$ with finite area $|\Omega|$ we have*

$$\int_\Omega \left[\rho^{\frac{1}{2}} - \left(\frac{\int_\Omega \rho}{|\Omega|} \right)^{\frac{1}{2}} \right]_+^4 \geq (1 - 4\epsilon) \int_\Omega \rho^2 + (2 - \epsilon^{-1}) \frac{(\int_\Omega \rho)^2}{|\Omega|}, \quad (30)$$

for arbitrary $\epsilon > 0$.

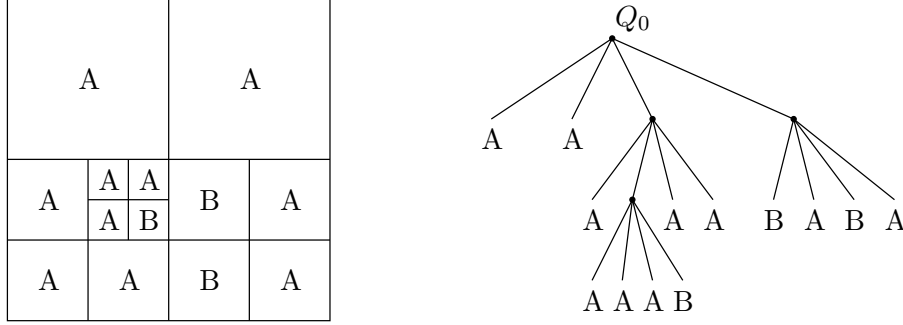


Figure 3: Example of a splitting of Q_0 and a corresponding tree \mathbb{T} of sub-squares. For the B-square at level 3 in the tree, the set $\mathcal{A}(Q)$ of associated A-squares consists of 8 elements, while for the two B-squares at level 2, $\mathcal{A}(Q)$ coincide and has 4 elements.

Proof. First note that

$$\int_{\Omega} \left[\rho^{\frac{1}{2}} - \left(\frac{\int_{\Omega} \rho}{|\Omega|} \right)^{\frac{1}{2}} \right]_+^4 \geq \int_{\Omega} \left[\rho^{\frac{1}{2}} - \left(\frac{\int_{\Omega} \rho}{|\Omega|} \right)^{\frac{1}{2}} \right]^4 - \frac{(\int_{\Omega} \rho)^2}{|\Omega|},$$

where the first integral on the r.h.s. is equal to

$$\int_{\Omega} \rho^2 - 4 \int_{\Omega} \rho^{\frac{1}{2}} \left(\frac{\int_{\Omega} \rho}{|\Omega|} \right)^{\frac{3}{2}} - 4 \int_{\Omega} \rho^{\frac{3}{2}} \left(\frac{\int_{\Omega} \rho}{|\Omega|} \right)^{\frac{1}{2}} + 7 \frac{(\int_{\Omega} \rho)^2}{|\Omega|}.$$

Together with Hölder's inequality applied to the negative terms,

$$\int_{\Omega} \rho^{\frac{1}{2}} \leq \left(\int_{\Omega} \rho \right)^{\frac{1}{2}} |\Omega|^{\frac{1}{2}}, \quad \text{and} \quad \int_{\Omega} \rho \cdot \rho^{\frac{1}{2}} \left(\frac{\int_{\Omega} \rho}{|\Omega|} \right)^{\frac{1}{2}} \leq \epsilon \int_{\Omega} \rho^2 + \frac{1}{4\epsilon} \frac{(\int_{\Omega} \rho)^2}{|\Omega|},$$

this produces the inequality (30). \square

Theorem 11 (Kinetic energy inequality for anyons). *Let $u \in \mathcal{D}_{0,\alpha}^N$ be an N -anyon wavefunction on \mathbb{R}^2 , with $N \geq 2$. Then*

$$\sum_{j=1}^N \int_{\mathbb{R}^{2N}} |D_j u|^2 dx \geq C_K C_{\alpha,N}^2 \int_{\mathbb{R}^2} \rho(\mathbf{x})^2 d\mathbf{x}, \quad (31)$$

for some positive constant C_K .

Proof. Given u , and hence ρ , smooth and supported on some square $Q_0 \subseteq \mathbb{R}^2$, we split up the domain $Q := Q_0$ into ever smaller subsquares according to the following algorithm (see Figure 3):

- A given square Q is split into four subsquares $Q'_j \in \{1,2,3,4\}$ s.t. $|Q'_j| = |Q|/4$.
- Whenever $\int_{Q'_j} \rho < 2$, we will not split that square Q'_j further, and we mark it A.
- If all four squares Q'_j are marked A, then we back up to the bigger square Q and mark it B. (Thus, we then stop splitting Q and the subsquares Q'_j are discarded.)
- For each of the unmarked squares Q'_j (i.e. those for which $\int_{Q'_j} \rho \geq 2$) we iterate the splitting algorithm with $Q := Q'_j$.

In the end (note that the procedure will eventually stop since ρ is integrable), we will have a system of subsquares Q organized in a tree \mathbb{T} , such that:

- The whole domain Q_0 is covered by squares of type A or B.
- Let us denote, for a given B-square Q_B ,

$$\mathcal{A}(Q_B) := \left\{ \begin{array}{l} \text{All A-squares } Q_A \in \mathbb{T} \text{ that can be found by going} \\ \text{back in the tree from } Q_B \text{ (possibly all the way to} \\ Q_0) \text{ and then one step forward.} \end{array} \right\},$$

then every A-square $Q_A \in \mathbb{T}$ satisfies $Q_A \in \mathcal{A}(Q_B)$ for some B-square $Q_B \in \mathbb{T}$. (Note that at least one B-square can be found among the leaves in the highest level of every branch of the tree.)

- $0 \leq \int_{Q_A} \rho < 2$ for every A-square Q_A .
- $2 \leq \int_{Q_B} \rho < 8$ for every B-square Q_B .

Finally, we also divide the A-squares into a subclass \mathcal{A}_1 for those subsquares on which ρ is approximately constant, and \mathcal{A}_2 for those subsquares with a non-constant density:

$$\mathcal{A}_1 := \left\{ \text{all A-squares } Q \in \mathbb{T} \text{ s.t. } \int_Q \rho^2 \leq c \frac{(\int_Q \rho)^2}{|Q|} \right\},$$

$$\mathcal{A}_2 := \left\{ \text{all A-squares } Q \in \mathbb{T} \text{ s.t. } \int_Q \rho^2 > c \frac{(\int_Q \rho)^2}{|Q|} \right\},$$

for some fixed constant $c > 1$, to be chosen below. We will then split the kinetic energy integral T in the l.h.s. of (31) into a sum over all the marked subsquares $Q \in \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{B}$ forming the leaves of the tree \mathbb{T} , and consider the three different types of squares separately.

Let us first consider the kinetic energy T_{Q_B} on any B-square $Q = Q_B$. We split it up into two parts using $\kappa \in (0, 1)$ and apply both Lemma 9 (local uncertainty principle) and Lemma 8 (local Pauli principle) to conclude

$$\begin{aligned} T_{Q_B} &\geq \kappa \frac{C'_2}{\int_Q \rho} \int_Q \left[\rho^{\frac{1}{2}} - \left(\frac{\int_Q \rho}{|Q|} \right)^{\frac{1}{2}} \right]_+^4 + (1 - \kappa) \frac{c_Q C_{\alpha, N}^2}{|Q|} \left(\int_Q \rho - 1 \right) \\ &\geq \frac{\kappa C'_2}{8} (1 - 4\epsilon) \int_Q \rho^2 + \left(\frac{\kappa C'_2}{8} (2 - \epsilon^{-1}) + (1 - \kappa) \frac{7c_Q C_{\alpha, N}^2}{64} \right) \frac{(\int_Q \rho)^2}{|Q|}, \end{aligned}$$

where we also used that $x - 1 \geq \frac{7}{64}x^2$ for $2 \leq x < 8$, as well as Lemma 10. Choosing κ and ϵ appropriately, we conclude

$$T_{Q_B} \geq C_{\alpha, N}^2 \left(c_1 \int_{Q_B} \rho^2 + c_2 \frac{(\int_{Q_B} \rho)^2}{|Q_B|} \right), \quad (32)$$

with $c_1, c_2 > 0$.

For the A₂-squares $Q = Q_A$, we use that the energy given by the local uncertainty principle is large due to a sufficiently non-constant ρ :

$$T_{Q_A} \geq \frac{C'_2}{\int_Q \rho} \int_Q \left[\rho^{\frac{1}{2}} - \left(\frac{\int_Q \rho}{|Q|} \right)^{\frac{1}{2}} \right]_+^4 \geq \frac{C'_2}{2} (1 - 4\epsilon - (\epsilon^{-1} - 2)c^{-1}) \int_Q \rho^2,$$

for $0 < \epsilon < 1/4$, where we again used Lemmas 9 and 10. Taking $\epsilon := 1/8$ and $c := 24$ we find

$$T_{Q_A} \geq \frac{C'_2}{8} \int_{Q_A} \rho^2. \quad (33)$$

Lastly, we show that the remaining squares, of type A₁, have a negligible contribution to the energy compared to that already obtained from the Pauli energy on the B-squares. Namely, as observed above, every subsquare of type A₁ is contained in $\mathcal{A}_1(Q_B) := \mathcal{A}(Q_B) \cap \mathcal{A}_1$ for some B-square Q_B . Now, note that for each B-square $Q_B \in \mathbb{T}$, say at a level $k \in \mathbb{N}$ in the tree, we have

$$\frac{(\int_{Q_B} \rho)^2}{|Q_B|} \geq \frac{4}{4^{-k}|Q_0|},$$

while the total integral of ρ^2 over all A₁-squares associated with Q_B is at most

$$\sum_{Q \in \mathcal{A}_1(Q_B)} \int_Q \rho^2 \leq \sum_{j=1}^k \sum_{\substack{Q \in \mathcal{A}_1(Q_B) \\ \text{at level } j}} c \frac{(\int_Q \rho)^2}{|Q|} \leq \sum_{j=1}^k 3c \frac{4}{4^{-j}|Q_0|} \leq 96 \frac{4^{k+1}}{|Q_0|}.$$

Hence, by (32),

$$T_{Q_B} \geq C_{\alpha,N}^2 \left(c_1 \int_{Q_B} \rho^2 + \frac{c_2}{96} \sum_{Q \in \mathcal{A}_1(Q_B)} \int_Q \rho^2 \right). \quad (34)$$

Summing everything up, it follows from (33) and (34) that the total kinetic energy is

$$\begin{aligned} T &= \sum_{j=1}^N \int_{\mathbb{R}^{2N}} |D_j u|^2 \left(\sum_{Q_A \in \mathcal{A}} \chi_{Q_A}(\mathbf{x}_j) + \sum_{Q_B \in \mathcal{B}} \chi_{Q_B}(\mathbf{x}_j) \right) dx \\ &\geq \sum_{Q_A \in \mathcal{A}_2} T_{Q_A} + \sum_{Q_B \in \mathcal{B}} T_{Q_B} \geq C_K C_{\alpha,N}^2 \int_{Q_0} \rho^2, \end{aligned}$$

for a positive constant $C_K := \min\{c_1, c_2/96, C'_2/8\}$. \square

Corollary (Lieb-Thirring inequality for anyons). *Let $u \in \mathcal{D}_{0,\alpha}^N$ be an N -anyon wavefunction and V a real-valued potential on \mathbb{R}^2 . Then*

$$\sum_{j=1}^N \int_{\mathbb{R}^{2N}} (|D_j u|^2 + V(\mathbf{x}_j) |u|^2) dx \geq -C_{LT} C_{\alpha,N}^{-2} \int_{\mathbb{R}^2} |V_-(\mathbf{x})|^2 d\mathbf{x}, \quad (35)$$

for some positive constant C_{LT} .

Proof. The l.h.s. is bounded below by

$$C_K C_{\alpha,N}^2 \int_{\mathbb{R}^2} \rho^2 d\mathbf{x} - \int_{\mathbb{R}^2} |V_-| \rho d\mathbf{x} \geq -\frac{1}{4} C_K^{-1} C_{\alpha,N}^{-2} \int_{\mathbb{R}^2} V_-^2 d\mathbf{x},$$

where we used $\int |V_-| \rho \leq (\int V_-^2)^{\frac{1}{2}} (\int \rho^2)^{\frac{1}{2}}$ and minimized w.r.t. $\int \rho^2$. \square

Remark. Theorem 11 immediately implies the rough bound

$$T \geq C_K C_{\alpha,N}^2 \int_{\Omega} \rho^2 d\mathbf{x} \geq \frac{C_K C_{\alpha,N}^2}{|\Omega|} \left(\int_{\Omega} \rho \cdot 1 d\mathbf{x} \right)^2 = C_K C_{\alpha,N}^2 \frac{N^2}{|\Omega|} \quad (36)$$

for the ground state energy of a non-interacting gas of anyons supported on a domain $\Omega \subseteq \mathbb{R}^2$.

Appendix A: Improvements of the local energy

Here we give some comments on how the explicit bound for the energy in Lemma 7 could be improved. One alternative approach for taking the full many-particle domain Ω^N into account is to extend the Hardy inequality (18) using a variant of Theorem 2.2 in [1], which here again has been modified to account for the underlying symmetry.

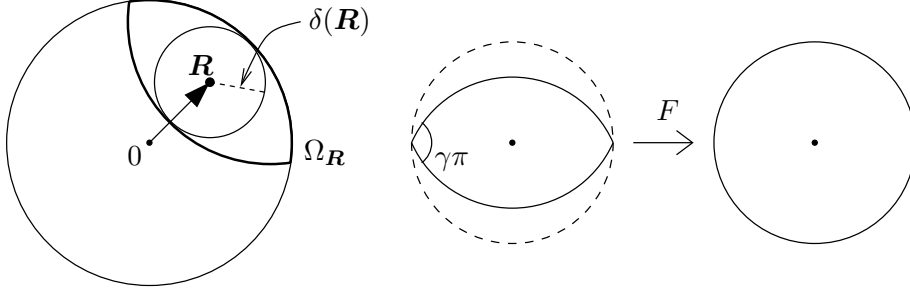


Figure 4: The set $\Omega_{\mathbf{R}}$, which can be mapped conformally to the unit disk by the map F .

Theorem 12 (Many-anyon Hardy on a disk). *Let $\Omega := B_\lambda(0)$ be a disk of radius λ in \mathbb{R}^2 and let $u \in \mathcal{D}_{0,\alpha}^N$. Then*

$$\int_{\Omega^N} \sum_{j=1}^N |D_j u|^2 dx \geq \frac{C_{\alpha,N}^2}{N\lambda^2} \int_{\Omega^N} \sum_{i < j} f\left(\frac{\mathbf{x}_i + \mathbf{x}_j}{2\lambda}, \frac{\mathbf{x}_i - \mathbf{x}_j}{2\lambda\sqrt{1-R^2}}\right) |u|^2 dx, \quad (37)$$

where

$$f(\mathbf{R}, \tilde{\mathbf{r}}) := \frac{16}{\gamma_R^2(1-R^2)} \left| (1+z)^{1+\frac{1}{\gamma_R}} (1-z)^{1-\frac{1}{\gamma_R}} - (1-z)^{1+\frac{1}{\gamma_R}} (1+z)^{1-\frac{1}{\gamma_R}} \right|^{-2},$$

z is the complexification of the renormalized coordinate $\tilde{\mathbf{r}}$ with imaginary axis given by the unit vector \mathbf{R}/R , and $\gamma_R := 1 - \frac{2}{\pi} \arcsin(R)$.

Proof. We proceed as in the proof of Theorem 4, but this time parameterize not just $\Omega \circ \Omega$, but the whole set Ω^2 by the relative coordinates \mathbf{R}, \mathbf{r} . For \mathbf{R} at a distance λR , $0 \leq R \leq 1$, from the center 0 of the disk, the total range of allowed \mathbf{r} forms an eye-shaped set $\Omega_{\mathbf{R}}$ given by the intersection of two disks (see Figure 4). After rescaling by $(1-R^2)^{-\frac{1}{2}}\lambda^{-1}$ and choosing axes so that the corners of the eye are at the points $z = -1$ resp. $z = 1$ in the complex plane, we can map the resulting set conformally onto the unit disk by the sequence of maps

$$F: \quad z \mapsto \frac{1+z}{1-z} =: w, \quad w \mapsto w^{\frac{1}{\gamma}} =: \zeta, \quad \zeta \mapsto \frac{\zeta-1}{\zeta+1} =: F(z),$$

where $\gamma\pi = \gamma_R\pi$ is the angle at the corner of the eye. The resulting map is hence

$$F(z) = \frac{(1+z)^{\frac{1}{\gamma}} - (1-z)^{\frac{1}{\gamma}}}{(1+z)^{\frac{1}{\gamma}} + (1-z)^{\frac{1}{\gamma}}} =: \xi + i\eta, \quad (38)$$

which is *antisymmetric* w.r.t. the antipodal map $z \mapsto -z$. Now, under the

coordinate transformation F , the integral w.r.t. \mathbf{r} in (21) becomes

$$\begin{aligned} \int_{\Omega_{\mathbf{R}}} |(D_j - D_k)u(\mathbf{r})|^2 d\mathbf{r} &= \int_{B_1(0)} |(D_{F(\tilde{\mathbf{r}})}v(\xi, \eta))^2 d\xi d\eta \\ &\geq C_{\alpha, N}^2 \int_{B_1(0)} \frac{|v|^2}{\xi^2 + \eta^2} d\xi d\eta = \frac{C_{\alpha, N}^2}{(1 - R^2)\lambda^2} \int_{\Omega_{\mathbf{R}}} \frac{|F'(z)|^2}{|F(z)|^2} |u|^2 d\mathbf{r}, \end{aligned}$$

where we again applied an annular decomposition of $B_1(0)$ and Lemma 2, and used the antipodal symmetry of $v(\xi, \eta) := u(F^{-1}(\xi + i\eta))$. Finally, we have by the definition of f that $|F'(z)|^2/|F(z)|^2 = (1 - R^2)f(\mathbf{R}, \tilde{\mathbf{r}})$. \square

Using the above Hardy potential f in (26), together with more precise estimates of the lowest eigenvalue of the corresponding operator H , and the restriction that the eigenfunction should be antipodal-symmetric, would almost certainly produce a significantly better bound on the constant c_Ω and hence on the energy of a gas of anyons.

Appendix B: A local Lieb-Thirring inequality with Neumann boundary conditions

In this appendix we derive certain bosonic and fermionic kinetic energy inequalities on domains with Neumann boundary conditions. These follow straightforwardly from a recent method due to Rumin [18].

Theorem 13. *Let Q be a cube in \mathbb{R}^d with volume $|Q|$. Then*

$$\sum_{j=1}^N \|\nabla \phi_j\|_{L^2(Q)}^2 \geq C'_d (\int_Q \rho)^{-\frac{2}{d}} \int_Q \left[\rho(\mathbf{x})^{\frac{1}{2}} - \left(\frac{\int_Q \rho}{|Q|} \right)^{\frac{1}{2}} \right]_+^{\frac{2(d+2)}{d}} d\mathbf{x}, \quad (39)$$

where $\phi_j \in H^1(Q)$ and $\rho(\mathbf{x}) := \sum_{j=1}^N |\phi_j(\mathbf{x})|^2$.

Moreover, if $\{\phi_j\}$ are orthonormal in $L^2(Q)$, then

$$\sum_{j=1}^N \|\nabla \phi_j\|_{L^2(Q)}^2 \geq C'_d \int_Q \left[\rho(\mathbf{x})^{\frac{1}{2}} - |Q|^{-\frac{1}{2}} \right]_+^{\frac{2(d+2)}{d}} d\mathbf{x}. \quad (40)$$

Proof. For any $e \geq 0$ and $\phi \in L^2(Q)$, let

$$\phi = \phi^{e,+} + \phi^{e,-}, \quad \phi^{e,+} := P_{\{-\Delta_Q^{\mathcal{N}} \geq e\}} \phi, \quad \phi^{e,-} := P_{\{-\Delta_Q^{\mathcal{N}} < e\}} \phi,$$

and note that for $\phi \in H^1(Q)$ (interpreted in terms of quadratic forms)

$$\begin{aligned} \int_0^\infty \|\phi^{e,+}\|_{L^2(Q)}^2 de &= \langle \phi, \int_0^\infty \underbrace{P_{\{-\Delta_Q^{\mathcal{N}} \geq e\}}}_{=\int_{\mathbb{R}} 1_{\{\lambda \geq e\}} dP(\lambda)} de \phi \rangle = \langle \phi, \underbrace{-\Delta_Q^{\mathcal{N}}}_{=\int_{\mathbb{R}} \lambda dP(\lambda)} \phi \rangle. \end{aligned}$$

Denote the eigenvalues and orthonormal eigenfunctions of $-\Delta_Q^{\mathcal{N}}$ by $\{\lambda_k\}_{k=0}^{\infty}$ resp. $\{u_k\}_{k=0}^{\infty}$. Then for each $\mathbf{x} \in Q$,

$$\begin{aligned} |\phi^{e,-}(\mathbf{x})|^2 &= \left| \left(P_{\{-\Delta_Q^{\mathcal{N}} < e\}} \phi \right) (\mathbf{x}) \right|^2 = \left| \sum_{\lambda_k < e} \langle u_k, \phi \rangle u_k(\mathbf{x}) \right|^2 \\ &= \left| \left\langle \sum_{\lambda_k < e} \overline{u_k(\mathbf{x})} u_k, \phi \right\rangle \right|^2 \leq \left(\sum_{\lambda_k < e} |u_k(\mathbf{x})|^2 \right) \|\phi\|^2 \\ &\leq \left(\frac{1}{|Q|} + \sum_{0 < \lambda_k < e} \frac{2^d}{|Q|} \right) \|\phi\|^2, \quad (41) \end{aligned}$$

and hence $|\phi^{e,-}(\mathbf{x})|^2 \leq (|Q|^{-1} + C_d e^{\frac{d}{2}}) \|\phi\|^2$ by the well-known asymptotics for the Neumann eigenvalues on Q .

Now, by the triangle inequality in \mathbb{C}^N we have for arbitrary $\{\phi_j\}_{j=1}^N \subseteq L^2(Q)$

$$\left(\sum_{j=1}^N |\phi_j^{e,+}(\mathbf{x})|^2 \right)^{\frac{1}{2}} \geq \left[\left(\sum_{j=1}^N |\phi_j(\mathbf{x})|^2 \right)^{\frac{1}{2}} - \left(\sum_{j=1}^N |\phi_j^{e,-}(\mathbf{x})|^2 \right)^{\frac{1}{2}} \right]_+, \quad \mathbf{x} \in Q,$$

and hence we find that for $\rho(\mathbf{x}) := \sum_j |\phi_j(\mathbf{x})|^2$ and $\{\phi_j\}_{j=1}^N \subseteq H^1(Q)$

$$\begin{aligned} \sum_{j=1}^N \|\nabla \phi_j\|^2 &= \sum_j \int_Q \int_0^\infty |\phi_j^{e,+}(\mathbf{x})|^2 de d\mathbf{x} \\ &\geq \int_Q \int_0^\infty \left[\rho(\mathbf{x})^{\frac{1}{2}} - \left((|Q|^{-1} + C_d e^{\frac{d}{2}}) \int_Q \rho(\mathbf{y}) d\mathbf{y} \right)^{\frac{1}{2}} \right]_+^2 de d\mathbf{x} \\ &\geq C_d \int_Q \rho(\mathbf{y}) d\mathbf{y} \int_Q \int_0^\infty \left[\left(\frac{\rho(\mathbf{x})}{C_d \int_Q \rho} \right)^{\frac{1}{2}} - \left(\frac{1}{C_d |Q|} \right)^{\frac{1}{2}} - e^{\frac{d}{4}} \right]_+^2 de d\mathbf{x} \\ &= \frac{d^2 C_d \int_Q \rho}{(d+2)(d+4)} \int_Q \left[\left(\frac{\rho(\mathbf{x})}{C_d \int_Q \rho} \right)^{\frac{1}{2}} - \left(\frac{1}{C_d |Q|} \right)^{\frac{1}{2}} \right]_+^{\frac{2(d+2)}{d}} d\mathbf{x} \\ &= C'_d (\int_Q \rho)^{1-\frac{d+2}{d}} \int_Q \left[\rho(\mathbf{x})^{\frac{1}{2}} - \left(\frac{\int_Q \rho}{|Q|} \right)^{\frac{1}{2}} \right]_+^{\frac{2(d+2)}{d}} d\mathbf{x}, \quad (42) \end{aligned}$$

with $C'_d := d^2 C_d^{-\frac{2}{d}} / (d+2)(d+4)$.

In the case that $\{\phi_j\}$ are orthonormal, Bessel's inequality applies in (41):

$$\sum_j \left| \left\langle \sum_{\lambda_k < e} \overline{u_k(\mathbf{x})} u_k, \phi_j \right\rangle \right|^2 \leq \left\| \sum_{\lambda_k < e} \overline{u_k(\mathbf{x})} u_k \right\|^2 = \sum_{\lambda_k < e} |u_k(\mathbf{x})|^2,$$

which then replaces every occurrence of $\int_Q \rho$ in (42) by 1. \square

Remark (Generalization). If we know how $\Xi_\Omega(e, \mathbf{x}) := \sum_{\lambda_k < e} |u_k(\mathbf{x})|^2$ behaves for a given domain $\Omega \subseteq \mathbb{R}^d$, then we can evaluate

$$\sum_{j=1}^N \langle \phi_j, -\Delta_\Omega \phi_j \rangle \geq \int_\Omega \int_0^\infty \left[\rho(\mathbf{x})^{\frac{1}{2}} - \left(\Xi_\Omega(e, \mathbf{x}) \int_\Omega \rho \right)^{\frac{1}{2}} \right]_+^2 de d\mathbf{x} \quad (43)$$

for either the Neumann or Dirichlet Laplacian on Ω . Again, $\int_\Omega \rho$ in the r.h.s. is replaced by 1 in the case that $\{\phi_j\}$ are orthonormal.

Theorem 14 (Many-particle version). *Let Q be a cube in \mathbb{R}^d with volume $|Q|$, and let $u \in H^1(\mathbb{R}^{dN})$ be an N -particle wavefunction. Then*

$$\sum_{j=1}^N \int_{\mathbb{R}^{dN}} |\nabla_j u|^2 \chi_Q(\mathbf{x}_j) d\mathbf{x} \geq C'_d (\int_Q \rho)^{-\frac{2}{d}} \int_Q \left[\rho(\mathbf{x})^{\frac{1}{2}} - \left(\frac{\int_Q \rho}{|Q|} \right)^{\frac{1}{2}} \right]_+^{\frac{2(d+2)}{d}} d\mathbf{x}, \quad (44)$$

where $\rho(\mathbf{x}) := \sum_{j=1}^N \int_{\mathbb{R}^{d(N-1)}} |u(\mathbf{x}_1, \dots, \mathbf{x}_{j-1}, \mathbf{x}, \mathbf{x}_{j+1}, \dots, \mathbf{x}_N)|^2 \prod_{k \neq j} d\mathbf{x}_k$.

Proof. We define for each $x' = (\mathbf{x}_1, \dots, \mathbf{x}_{j-1}, \mathbf{x}_{j+1}, \dots, \mathbf{x}_N) \in \mathbb{R}^{d(N-1)}$ a collection of functions

$$\mathbf{x} \mapsto \phi_j(\mathbf{x}, x') := u(\mathbf{x}_1, \dots, \mathbf{x}_{j-1}, \mathbf{x}, \mathbf{x}_{j+1}, \dots, \mathbf{x}_N)$$

in $L^2(Q)$, and proceed as in the proof of Theorem 13, writing

$$\int_Q |\nabla_{\mathbf{x}} \phi_j(\mathbf{x}, x')|^2 d\mathbf{x} = \int_0^\infty \int_Q |\phi_j^{e,+}(\mathbf{x}, x')|^2 d\mathbf{x} de,$$

for each $x' \in \mathbb{R}^{d(N-1)}$, and similarly to (41)

$$\begin{aligned} & \sum_{j=1}^N \int_{\mathbb{R}^{d(N-1)}} |\phi_j^{e,-}(\mathbf{x}, x')|^2 dx' \\ & \leq \sum_{j=1}^N \int_{\mathbb{R}^{d(N-1)}} \left(|Q|^{-1} + C_d e^{\frac{d}{2}} \right) \int_Q |\phi_j(\mathbf{y}, x')|^2 d\mathbf{y} dx' \\ & = \left(|Q|^{-1} + C_d e^{\frac{d}{2}} \right) \int_Q \rho, \end{aligned}$$

as well as using the triangle inequality on $L^2(\mathbb{R}^{d(N-1)}; \mathbb{C}^N)$,

$$\left(\int_{\mathbb{R}^{d(N-1)}} \sum_{j=1}^N |\phi_j^{e,+}(\mathbf{x}, x')|^2 dx' \right)^{\frac{1}{2}} \geq \left[\left(\int_{\mathbb{R}^{d(N-1)}} \sum_{j=1}^N |\phi_j(\mathbf{x}, x')|^2 dx' \right)^{\frac{1}{2}} - \left(\int_{\mathbb{R}^{d(N-1)}} \sum_{j=1}^N |\phi_j^{e,-}(\mathbf{x}, x')|^2 dx' \right)^{\frac{1}{2}} \right]_+,$$

for $\mathbf{x} \in Q$. Hence,

$$\begin{aligned} \sum_{j=1}^N \int_{\mathbb{R}^{d(N-1)}} \int_Q |\nabla_{\mathbf{x}} \phi_j(\mathbf{x}, x')|^2 d\mathbf{x} dx' \\ \geq \int_Q \int_0^\infty \left[\rho(\mathbf{x})^{\frac{1}{2}} - \left((|Q|^{-1} + C_d e^{\frac{d}{2}}) \int_Q \rho \right)^{\frac{1}{2}} \right]_+^2 de d\mathbf{x}, \end{aligned}$$

and (44) then follows as in (42). \square

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